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1985 J. Phys. A: Math. Gen. 18 749

(http://iopscience.iop.org/0305-4470/18/5/009)

J. Phys. A: Math. Gen. 18 (1985) 749-764. Printed in Great Britain

On the exact reduction of a univariate catastrophe to normal form†

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Received 16 July 1984

Abstract. Quantitative applications of elementary catastrophe theory require exact determination of the equivalence transformations involved. Let $\phi(s; c)$ be an unfolding (which need not be universal) in $c \in \mathbb{R}^K$ of a cuspoid singularity A_k in $s \in \mathbb{R}$. We discuss its reduction via a sequence of coordinate transformations to normal form, exact to degree m in the control variables c, and show that this requires knowledge of the terms of ϕ only to degrees l in c and j in s satisfying $(l-m-1)k+j+1 \leq 0$. The 'linear normal form', which describes the orientation and shear of the bifurcation set, is discussed in detail, and normal form methods for deriving tangent spaces and curvatures of singularity manifolds are described, with examples.

1. Introduction

An important application of elementary catastrophe theory is in determining not only the topological, but also the *metrical*, behaviour of a system by relating it to the appropriate normal form. The simpler normal forms have now been studied in some detail—for the general background see e.g. Poston and Stewart (1978). A particularly good example of this type of application is the asymptotic evaluation of oscillatory integrals by 'uniform approximation' (Duistermaat 1974, Berry 1976, Connor 1976). Such applications require that part, at least, of the transformation to normal form be determined in some sense exactly. For the uniform approximation problem some fairly elaborate exact methods have been developed by quantum chemists (Uzer and Child 1982, Connor *et al* 1984) with a view to numerical evaluation of the control-space mapping. We are concerned with the algebraic aspect of the transformation, and the analysis presented here arose in connection with a study of caustics generated by line sources or edges of wavefronts (Dangelmayr and Wright 1985), in which some of the caustic geometry is determined by relating it to the canonical bifurcation geometry.

Most of the discussions of reduction to normal form in the physical literature tend to involve *ad hoc* reductions for specific problems, and those in the mathematical literature are concerned with existence proofs or with singularities only. The *existence* of transformations to normal form was first proved by Mather (1968), but we are not aware of any previously published algorithms for constructing the transformations. A constructive proof of the splitting lemma was given by Gromoll and Meyer (1969) in

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a Hilbert-space formulation, and was presented in a finite-dimensional form by Poston and Stewart (1978), but this does not give a reduction to normal form. An explicit algorithm derived from the present analysis, which we believe should be easily and directly implementable using computer algebra, is given by Wright and Dangelmayr (1985).

We shall restrict our attention to the simplest case—unfoldings of singularities in univariate functions . Specifically, let $s \in \mathbb{R}$ be the physical state variable, $c \in \mathbb{R}^K$ the physical control variable (vector), and suppose that $\phi(s; c)$ is an unfolding of an A_k singularity (Arnol'd 1975, see also (1) below for a definition) at $s=0$, $c=0$. The problem is to find a transformation $s \to s'(s; c)$, whose existence is guaranteed by the general theory, that reduces ϕ to the normal form

$$\phi(s; c) = \phi'(s'; c) = \lambda(c) + \sum_{j=1}^{k-1} u_j(c) s'^j \pm s'^{k+1}$$

and thereby generates the 'unfolding functions' $u_j(c)$, which must satisfy $u_j(0)=0$. The complete 'right-equivalence' that transforms $\phi(s; c)$ into normal form is the set of maps

$$s'(s; c), \{u_j(c)\}_{j=1}^{k-1}, \lambda(c),$$

of which all but $\lambda(c)$ must satisfy certain rank conditions (see e.g. Wassermann (1974) for the versal case). Although we shall not do so here, it is not hard to show that the construction given below does satisfy the rank conditions. (This has been proved in detail by Karl Millington, private communication, for the specific algorithm given by Wright and Dangelmayr (1985).) We do not assume $\phi(s; c)$ to be a universal or even a versal (stable) unfolding; hence we do not require $K = k-1$ (see especially § 8).

Most commonly in practice one is interested only in the unfolding functions $u_j(c)$, and perhaps also in the shift-term $\lambda(c)$, so that the transformation $s'(s; c)$ need be determined only as far as necessary to find these functions. This point of view is implicit in our analysis. Generally, it is impossible to find any of these functions in closed form, so one has to settle for their Taylor polynomials in c to some finite degree m. We assume that it is known in advance what is a sufficient value of m (often this will be 1 or 2). We return to the important special case $m=1$ in § 7.

In §§ 2-4 we decompose the reduction process into a number of simple steps, each of which has a specific effect in relation to the normal form. We find this decomposition to be the best way to gain an understanding of the reduction process, and we have also found it to be convenient to implement by hand in simple cases. For example, in our line source study (Dangelmayr and Wright 1985) we needed to determine all linear terms, plus some particular quadratic terms that determine the caustic curvatures, in the reductions to normal form of fold, cusp and swallowtail catastrophes.

A major result, derived in § 5, is that in order to determine the normal form to degree m in c, it is necessary and sufficient to know the terms of ϕ to maximum degree $(m+1)k-1$ in s; specifically to degrees j in s and l in c satisfying $(l-m-1)k+j+1 \leq 0$. Hence to determine even the complete linear behaviour of the unfolding it is necessary to consider all terms up to degree $(2k-1)$ in s. This is despite the fact that an A_k singularity is $(k+1)$-determined, which emphasises that determinacy applies only to singularities, and not to unfoldings.

In § 6 we mention alternative algorithms and uniqueness of the reduction, and analyse the degree to which the state-space mapping $s \to s'(s; c)$ must (in principle) be determined in order to determine the normal form to degree m in c (and hence the degree to which it is available as a by-product of a reduction algorithm). Sections 7 and 8 provide examples of 'metrical catastrophe theory'. In § 7 the 'linear normal form' and the shearing effect of the $O(s^{k+2})$ 'tail' of the unfolding (defined in (1) below) are analysed in detail, and illustrated by an example calculation of the evolute of a simple plane curve. In § 8 we present general techniques, based on reduction to normal form, for determining tangent spaces and curvatures of singularity manifolds, or specific strata of bifurcation sets, such as ribs (cusped edges) in 3-space. We give two example calculations, the second of which actually involves two state variables and hence the splitting lemma.

2. Reduction to normal form in practice

We shall make considerable use of the terms *order* and *degree*, which we define as follows. For $x \in \mathbb{R}^n$, let $O(x^p)$, $H(x^p)$, $D(x^p)$ respectively denote any function of order p, homogeneous degree p, degree p in x; that is, any function depending on x only through terms of the form

$$\prod_{i=1}^{n} x_i^{p_i}, \qquad \text{where } \sum_{i=1}^{n} p_i \geq p,\ \sum_{i=1}^{n} p_i = p,\ \sum_{i=1}^{n} p_i \leq p$$

respectively and x_i is the ith component of $x \in \mathbb{R}^n$. For $n = 1$ these definitions take their familiar standard forms.

The starting point for the reduction is the Taylor expansion of ϕ about $s = 0$; we name the body and tail segments of this unfolding as shown in (1):

$$\phi(s;c) = \phi_0(c) + \underbrace{\sum_{j=1}^{k} \phi_j(c)s^j}_{\text{body}} + \phi_{k+1}(c)s^{k+1} + \underbrace{\sum_{j=k+2}^{\infty} \phi_j(c)s^j}_{\text{tail}}. \tag{1}$$

To have an A_k singularity at $s = 0$, $c = 0$, as we shall assume, the Taylor coefficients must satisfy

$$\phi_j(0) = 0 \text{ for } 1 \leq j \leq k \text{ and } \phi_{k+1}(0) \neq 0.$$

The practical restatement of the reduction problem that we study is to reduce ϕ to the form

$$\phi(s;c) = \phi'(s';c) = \lambda(c) + \sum_{j=1}^{k-1} u_j(c)s'^j \pm s'^{k+1} + O(c^{m+1}). \tag{2}$$

A priori, it suffices to work throughout to $D(c^m)$. The reduction can be conveniently decomposed into a number of steps, which are illustrated in figure 1 (but ignore the stepped boundary until § 5), each of which has a specific effect. There are two major stages.

Stage 1. Remove the $O(s^{k+2})$ tail from (1). We perform this iteratively to successively higher orders in c until what is left ($O(c^m)$) can be simply discarded.

Stage 2. (a) Reduce $\phi_{k+1}(c)$ to ± 1 to rescaling s; (b) Remove $\phi_k(c)s^k$ by a shift of the form $s \to s + \alpha(c)$.

Once stage 1 has been performed, stage 2 cannot produce any new tail terms, so stages 1 and 2 must be performed in that order. Stages 2(a) and (b) may be performed in either order.

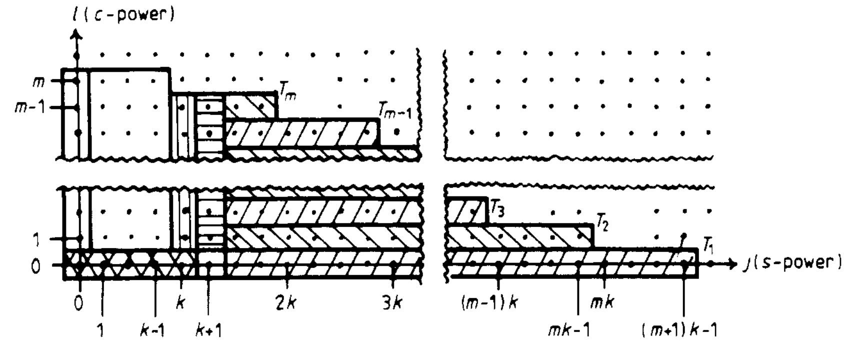

Figure 1. The 'term lattice': the point (j, l) represents $\phi_{j,l}$, the part of the Taylor coefficient ϕ_j of homogeneous degree l in c. The step-sided 'term triangle' shows those 'terms' that determine the normal form to degree m in c, where terms in the doubly hatched region are necessarily absent (i.e. zero). Any terms originating or produced outside this triangle may be simply discarded. Each diagonally hatched 'tongue' is removed by the stage 1 transformation shown. The horizontally hatched column with $j = k+1$ is removed by stage 2(a); the vertically hatched column with $j = k$ is removed by stage 2(b). The three unhatched areas left inside the term triangle represent the three distinct parts of the normal form, namely, from left to right,

$$\lambda(c), \quad \{u_j(c)s'^j\}_{j=1}^{k-1} \quad \text{and} \quad s'^{k+1}.$$

To avoid a notational catastrophe, we shall only keep track of the component transformations making up the complete reduction during our discussion of stage 1. Otherwise, we make only a very local distinction between any quantity and the result of transforming it. Thus at every step of the transformation we represent ϕ in the form (1), but allow the coefficient functions $\phi_j(c)$ to change until finally (1) takes the form (2); these different versions of ϕ are all right-equivalent.

3. Reduction stage 1

The tail $\Sigma_{j=k+2}^{\infty} \phi_j(c)s^j$ is removed from (1) by essentially absorbing it into the term $\phi_{k+1}(c)s^{k+1}$. If the body $\Sigma_{j=1}^{k} \phi_j(c)s^j$ of (1) were absent, this could be immediately accomplished by simply defining a transformation $s \to s'$ implicitly by

$$\phi_{k+1}(c)s'^{k+1} = \sum_{j=k+1}^{\infty} \phi_j(c)s^j. \tag{3}$$

Explicitly, this transformation has the form

$$s = s' + \sum_{i=2}^{\infty} t_i(c)s'^i. \tag{4}$$

Typically $\phi_j(c) = O(1)$ for $j \geq k+1$, so that also $t_i(c) = O(1)$ for all i.

However, if transformation (4) is applied to the unfolding body in (1), it generates a new tail. But because every s^j, $1 \leqslant j \leqslant k$, in the unfolding body is multiplied by $\phi_j(c) = O(c)$, this new tail is $O(c)$. Thus the effect of applying (4), determined solely by the tail of (1), to the whole of (1) is to replace the $O(1)$ tail by an $O(c)$ tail. The transformation (4) that would remove this $O(c)$ tail in the absence of an unfolding body will then have $t_i(c) = O(c)$, and will generate from the unfolding body a new tail that is $O(c^2)$. Clearly, the tail can be removed to any desired degree in c by iterating this process.

Let us denote the nth iteration of transformation (4), which transforms the $O(c^{n-1})$ tail into an $O(c^n)$ tail, by

$$T_n\colon s_{n-1} = s_n + \sum_{i=2}^{\infty} t_{i,n-1}(c) s_n^i \tag{5}$$

where $s_0 \equiv s$, and let us denote the result of applying T_n by $\phi^{(n)}(s_n; c)$, whose tail is $O(c^n)$. Then $T_n \phi^{(n-1)}(s_{n-1}; c) = \phi^{(n)}(s_n; c)$, where $\phi^{(0)} \equiv \phi$. The terms removed by each T_n are illustrated in figure 1. The coefficients $t_{i,n-1}(c)$ are $O(c^{n-1})$, but since T_n only removes the $H(c^{n-1})$ terms of the tail it suffices to make $t_{i,n-1}(c) = H(c^{n-1})$ and define T_n *solely* in terms of the $H(c^{n-1})$ part of the tail of $\phi^{(n-1)}$. Let us define $\phi_{j,l}^{(n)}(c)$ to be the $H(c^l)$ part of $\phi_j^{(n)}(c)$.

Although we do not presently need an explicit representation for T_n, it is interesting to see how easily T_n may be constructed. For $n \geqslant 2$, T_n may be defined by requiring

$$\phi_{k+1,0} s_{n-1}^{k+1} + \sum_{j=k+2}^{\infty} \phi_{j,n-1}(c) s_{n-1}^j = \phi_{k+1,0} s_n^{k+1} + O(c^n). \tag{6}$$

From (5),

$$s_{n-1}^j = s_n^j + j s_n^{j-1} \sum_{i=2}^{\infty} t_{i,n-1}(c) s_n^i + O(c^{2(n-1)}).$$

Substituting this into (6) gives

$$\phi_{k+1,0}\left(s_n^{k+1} + (k+1) s_n^k \sum_{i=2}^{\infty} t_{i,n-1}(c) s_n^i \right) + \sum_{j=k+2}^{\infty} \phi_{j,n-1}(c) s_n^j + O(c^{2(n-1)})$$
$$= \phi_{k+1,0} s_n^{k+1} + O(c^n).$$

By equating coefficients of s_n^i for $n \geqslant 2$ the coefficient functions of T_n are determined explicitly to be simply

$$t_{i,n-1}(c) = -\phi_{k+i,n-1}(c)/(k+1)\phi_{k+1,0} \qquad (n \geqslant 2). \tag{7}$$

T_1 has to be treated slightly differently; an explicit implementation of T_n (slightly different from that above) for all n is given by Wright and Dangelmayr (1985) (see especially algorithm A in § 4).

4. Reduction stage 2

Suppose the tail of ϕ has been completely removed by stage 1, leaving

$$\phi(s; c) = \phi_0(c) + \sum_{j=1}^{k} \phi_j(c) s^j + \phi_{k+1}(c) s^{k+1}. \tag{8}$$

Stages 2(a) and (b) may now be performed in either order; for generality we treat each as though it is performed first. The terms that each removes are illustrated in figure 1.

Stage 2.(a). $\phi_{k+1}(c)$ is reduced to ± 1 by the transformation

$$s = |\phi_{k+1}(c)|^{1/(k+1)} s'. \tag{9a}$$

The coefficient of s' has a well defined series expansion in c because $\phi_{k+1}(0) \neq 0$ (by assumption). This transformation does not affect $\phi_0(c)$, so because $\phi_j(c) = \mathrm{O}(c)$ for $1 \leqslant j \leqslant k$, transformation (9$a$) and hence $\phi_{k+1}(c)$ are required for stage 2(a) only to $\mathrm{D}(c^{m-1})$ in order that (8) remain accurate to $\mathrm{D}(c^m)$. Hence (9a) may be implemented by expanding up to $\mathrm{D}(c^{m-1})$, and its effect is to multiply each $\phi_j(c)$, $1 \leqslant j \leqslant k$, by a factor $\mathrm{O}(1)$. This produces a nonlinear, orientation-preserving, differential scale change in the final unfolding functions $u_j(c)$.

Stage 2.(b). $\phi_k(c)s^k$ is removed by the transformation

$$s = s' + \alpha(c), \tag{9b}$$

where $\alpha(c)$ satisfies

$$(k+1)\phi_{k+1}(c)\alpha(c) + \phi_k(c) = 0. \tag{10}$$

(If stage 2(a) has already been performed than $\phi_{k+1}(c) = \pm 1$.) As $\phi_{k+1}(c) = \mathrm{O}(1)$ and $\phi_k(c) = \mathrm{O}(c)$, it follows that $\alpha(c) = \mathrm{O}(c)$. The effect of transformation (9b) is that

$$\phi_j(c) \to \phi_j'(c) = \sum_{i=j}^{k+1} \binom{i}{j} \phi_i(c)(\alpha(c))^{i-j} \qquad \forall\, 0 \leqslant j \leqslant k+1. \tag{11}$$

Hence $\phi_{k+1}(c)$ is unchanged, $\phi_k(c) \to 0$, and for all $0 \leqslant j \leqslant k-1$, $\phi_j \to \phi_j(c) + \mathrm{O}(c^2)$. In preparation for determining curvatures (cf § 8), we note that for $0 \leqslant j \leqslant k-2$, the quadratic additions to ϕ_j come from transforming the term of next highest degree (s^{j+1}), whereas quadratic additions to ϕ_{k-1} come from transforming *both* s^k and s^{k+1} since $\phi_{k+1}(c) = \mathrm{O}(1)$.

Transformation (11) must be implemented to $\mathrm{D}(c^m)$. Because $\alpha(c)$ contributes to the normal form via (11) either multiplied by $\phi_i(c) = \mathrm{O}(c)$ for $1 \leqslant i \leqslant k$ or raised to a power of at least 2 when multiplied by $\phi_{k+1}(c) = \mathrm{O}(1)$, it is required only to $\mathrm{D}(c^{m-1})$. Consequently $\phi_k(c)$ and $\phi_{k+1}(c)$ are required only to $\mathrm{D}(c^{m-1})$ in determining $\alpha(c)$. Furthermore, $\phi_k(c)$ contributes directly to the normal form only via (11), in which it is multiplied at least once by $\alpha(c) = \mathrm{O}(c)$, and so again is required only to $\mathrm{D}(c^{m-1})$.

5. Finite degrees

We now establish that, because the normal form is required only to finite degree m in c, the infinite series (5) representing each stage 1 transformation T_n may be truncated. This leads to the following theorem.

Theorem 1: Normal form degree theorem. For $lk \leqslant j < (l+1)k$ with $0 \leqslant l \leqslant m$, the normal form for $\phi(s; c)$ to $\mathrm{D}(c^m)$ is completely determined by the coefficient functions $\phi_j(c)$ to $\mathrm{D}(c^{m-l})$.

This important result is illustrated in figure 1. In particular, $\phi_j(c)$ is not required at all for $j \geq (m+1)k$, so that the Taylor expansion of ϕ is required only to $\mathrm{D}(s^{(m+1)k-1})$.

Proof. The key to establishing the proof is the following observations:

(a) For $1 \leq j \leq k$, $\phi_j^{(n)}(c)$ is $\mathrm{O}(c)$, $\phi_{k+1}^{(n)}(c)$ is $\mathrm{O}(1)$ and, for $j \geq k+2$, $\phi_j^{(n)}(c)$ is $\mathrm{O}(c^n)$.

(b) T_n applied to s_{n-1}^j generates *new* terms that are $\mathrm{O}(s_n^{j+1})$ only, and T_n is determined only by the tail of $\phi^{(n-1)}$.

(c) Because T_n (cf (5)) is $\mathrm{O}(s_n)$, if $\phi_j^{(p)}(c)$ is required to $\mathrm{D}(c^l)$, then so is $\phi_j^{(q)}(c)$ for all $q \leq p$.

The sufficient degrees are best established by working backwards, and frequent reference to figure 1 should help in following the argument.

It was established in § 4 that stage 2 of the transformation requires ϕ_j to $\mathrm{D}(c^m)$ for $0 \leq j \leq k-1$ and to $\mathrm{D}(c^{m-1})$ for $k \leq j \leq k+1$, so this is what stage 1 must generate. T_{m+i}, $i \geq 1$, generates new terms that are $\mathrm{O}(c^{m+i})$ only (cf (5), § 3), so the last transformation that need be applied is T_m. T_m generates new terms that are $\mathrm{O}(c^m)$, but $\phi_k^{(m)}$ and $\phi_{k+1}^{(m)}$ are required only to $\mathrm{D}(c^{m-1})$. Also the tail of $\phi^{(m)}$ is not required at all because T_{m+1} is not required. Therefore, the effect of T_m is required only to $\mathrm{D}(s^{k-1})$, so T_m itself is required only to $\mathrm{D}(s^{k-1})$. This is the crucial observation, and its effect propagates.

We have deduced that the polynomial transformation

$$T_m\colon s_{m-1} = s_m + \sum_{i=2}^{k-1} t_{i,m-1}(c)s_m^i$$

suffices, where $t_{i,m-1}(c) = \mathrm{H}(c^{m-1})$. The $(k-2)$ coefficient functions $t_{i,m-1}(c)$ in T_m are determined (as in (7) for $m \geq 2$) solely by the $\mathrm{H}(c^{m-1})$ parts $\phi_{j,m-1}^{(m-1)}(c)$ of the first $(k-2)$ coefficient functions in the tail

$$\sum_{j=k+2}^{2k-1} \phi_j^{(m-1)}(c)s_{m-1}^j$$

of $\phi^{(m-1)}$. Therefore, for $k+2 \leq j \leq 2k-1$, $\phi_j^{(m-1)}(c)$, and hence all $\phi_j^{(n)}(c)$ with $n \leq m-1$, are required to $\mathrm{D}(c^{m-1})$, as shown in figure 1.

Since the $\mathrm{O}(c^{m-1})$ part of $\phi^{(m-1)}$ is required only to $\mathrm{D}(s^{2k-1})$, so is T_{m-1}. Thus the polynomial transformation

$$T_{m-1}\colon s_{m-2} = s_{m-1} + \sum_{i=2}^{2k-1} t_{i,m-2}(c)s_{m-1}^i$$

suffices, where $t_{i,m-2}(c) = \mathrm{H}(c^{m-2})$. The $(2k-2)$ coefficient functions $t_{i,m-2}(c)$ are determined by the $\mathrm{H}(c^{m-2})$ parts $\phi_{j,m-2}^{(m-2)}(c)$ of the first $(2k-2)$ coefficient functions in the tail

$$\sum_{j=k+2}^{3k-1} \phi_j^{(m-2)}(c)s_{m-2}^j$$

of $\phi^{(m-2)}$. Therefore, for $k+2 \leq j \leq 3k-1$, $\phi_j^{(m-2)}(c)$, and hence all $\phi_j^{(n)}(c)$ with $n \leq m-2$, are required to $\mathrm{D}(c^{m-2})$, (although for $k+2 \leq j \leq 2k-1$ these coefficients are already required to $\mathrm{D}(c^{m-1})$) as shown in figure 1.

Iterating this argument shows that for general $1 \leq n \leq m$, the polynomial transformation

$$T_n\colon s_{n-1} = s_n + \sum_{i=2}^{(m+1-n)k-1} t_{i,n-1}(c)s_n^i$$

suffices, determined solely by the tail terms

$$\sum_{j=k+2}^{(m-n+2)k-1} \phi_{j,n-1}^{(n-1)} s_{n-1}^{j}.$$

This proves the theorem.

In particular,

$$T_1\colon s = s_1 + \sum_{i=2}^{mk-1} t_{i,0}(c) s_1^i$$

suffices, determined by

$$\sum_{j=k+2}^{(m+1)k-1} \phi_{j,0} s^j,$$

which proves that ϕ is required only to $\mathrm{D}(s^{(m+1)k-1})$ as claimed (see figure 1).

In applying these transformations, any terms generated outside the 'term triangle' shown in figure 1 should be discarded. The overall transformation that removes the tail from (1) *in the presence of the unfolding body* is the composition $T = T_m \circ T_{m-1} \circ \ldots \circ T_1$ with unnecessary terms discarded. It must have the form

$$T\colon s = s' + \sum_{i=2}^{mk-1} t_i(c) s'^i \qquad (12)$$

where generally $t_i(c) = \mathrm{O}(1)$ for all i and, for $2 \leqslant lk \leqslant i < (l+1)k$ with $0 \leqslant l \leqslant m-1$, $t_i(c)$ is required to $\mathrm{D}(c^{m-l-1})$. Clearly T has no effect on $\phi_0(c)$.

6. Discussion

6.1. Implementation and alternative reduction algorithms

There are many equivalent ways of implementing an iterative reduction algorithm. The version we have discussed above breaks the complete reduction up into a number of steps, each of which has a specific effect: removing the tail to a specific degree in c, removing $\phi_k(c)$ or rescaling $\phi_{k+1}(c)$. The ultimate such decomposition gives an algorithm that removes one term from the term lattice of figure 1 at each iteration. This version of the algorithm is presented in Wright and Dangelmayr (1985); it gives an explicit and uniform algorithm for the whole reduction which is extremely simple to state. It also makes clear the extent to which the order of removal of terms is arbitrary, and presents an ordering that may be more convenient in practice than that which we have discussed here.

An alternative algorithm has been developed by Millington and Wright (1985), which makes a direct assault on the multivariate Taylor coefficients of the mappings involved. This very low level algorithm is much more complicated to specify and to implement, but may well be (much) more efficient, and hence allow more interesting problems to be solved in practice.

6.2. Uniqueness

Each transformation that was discussed in §§ 3, 4 preserved the sign of s. Subject to the condition that this be so, each transformation is uniquely determined by the result it is required to produce (otherwise a lot of trivial non-uniqueness arises, most of which cancels out in the overall transformation), and is (locally) invertible. This suggests that the overall transformation to normal form is unique. However, the order in which successive coordinate transformations are applied is not unique—see our discussion of stage 2, and particularly that in Wright and Dangelmayr (1985). It seems easiest to prove that the reduction is unique (subject to preserving the sign of s) from the Taylor coefficient approach, as done by Millington and Wright (1985), so we will not pursue the matter here.

We are currently considering reduction to normal form of multivariate catastrophes, which involves the splitting lemma and/or higher corank singularities. One problem that immediately arises is that in this case the reduction is not unique—arbitrary parameters appear at every step. (This is similar to the non-uniqueness of unfolding monomials for singularities of corank $\geqslant 2$—for example, there are two different normal forms for the hyperbolic umbilic catastrophe in common use.)

6.3. *The complete transformation*

We have based our algorithms around one part of the reduction to normal form, the mapping of the control variables c_i to canonical unfolding variables u_j, because for many applications this is all that is required. We have established that the information necessary to determine this mapping to $\mathrm{D}(c^m)$ is as stated in theorem 1 and illustrated in figure 1. (Note that consequently ϕ is not required to be C^∞, but only differentiable to the degrees given in (11), in order that the reduction to normal form can formally be performed. However, the result may then not be meaningful.)

It is interesting to ask to what extent this information determines the mapping $s \to s'(s; c)$. The answer is given by the following theorem.

Theorem 2. Transformation degree theorem. In determining the normal form for $\phi(s; c)$ to $\mathrm{D}(c^m)$, the state-space mapping $s \to s'(s; c)$ must be determined to $\mathrm{D}(s^{mk-1})$ in the form

$$s' = \sum_{i=0}^{mk-1} t_i'(c)s^i$$

where, for $lk \leqslant i < (l+1)k$ with $0 \leqslant l \leqslant m-1$, $t_i'(c)$ is determined to $\mathrm{D}(c^{m-l-1})$.

This is, of course, closely related to the normal form degree theorem given in § 5, and its proof involves a slight extension of the proof of that theorem.

Proof. Consider stages 1 and 2 separately and note that $s'(s; c)$ is the composition of stages 2(a) and (b) with stage 1. The complete stage 1 transformation T has (from 12) the form

$$s = \sum_{i=1}^{\infty} t_i s'^i. \tag{13}$$

Writing its inverse as

$$s' = \sum_{j=1}^{\infty} t_j' s^j \tag{14}$$

and substituting (13) into (14) gives

$$s' = \sum_{j=1}^{\infty} t'_j \left(\sum_{i=1}^{\infty} t_i s'^i \right)^j = \sum_{p=1}^{\infty} \left(\sum_{j=1}^{p} t'_j \sum_{|i|=p} \prod_{q=1}^{j} t_{i_q} \right) s'^p$$

where

$$i_q \geqslant 1 \qquad \text{and} \qquad |i| \equiv \sum_{q=1}^{j} i_q = p. \tag{15}$$

Now equate coefficients of s'^p. Taking $p=1$ shows that $t'_1 = 1/t_1 = 1$ (from 12) as expected, and taking $p \geqslant 2$ gives

$$0 = \sum_{j=1}^{p} t'_j \sum_{|i|=p} \prod_{q=1}^{j} t_{i_q}.$$

Since $1 \leqslant j \leqslant p$, (15) implies that $i_q \leqslant p$, so that t'_p is determined by t'_j with $j<p$ and t_i with $i \leqslant p$. Hence $t'_p(c)$ is determined to the maximum degree to which $t_i(c)$ is known *for all* $i \leqslant p$, which is the degree to which $t_p(c)$ is known since the degree to which $t_i(c)$ is known decreases monotonically as i increases. Hence the inverse of stage 1 alone satisfies the theorem.

The transformations 4(a and b) corresponding to stages 2(a and b) respectively are trivially inverted and are determined to $\mathrm{D}(c^{m-1})$ by $\phi_k(c)$ and $\phi_{k+1}(c)$. Composing these transformations with (14) does not reduce the degree to which any term of (14) is determined, which is $\leqslant (m-1)$, and introduces non-trivial coefficients $t'_0(c)$ and $t'_1(c)$ which are determined to $\mathrm{D}(c^{m-1})$. Hence the theorem is proved.

7. The linear normal form: orientation and shear

By the 'linear normal form' we mean the normal form to $\mathrm{D}(c)$. This is clearly the most important special case, and indeed is often all that is required in a normal form analysis. For example, it completely determines the orientation and shear of a physical bifurcation set relative to its canonical counterpart. This approach has been used by Dangelmayr and Wright (1985) in analysing caustics from a line source, and essentially the inverse of our present approach was used by Nye and Hannay (1984) to analyse 'the orientations and (linear) distortions of caustics in geometrical optics' in general.

The linear normal form is determined solely by the truncated expansion

$$\phi(s;c) = \underbrace{\sum_{j=0}^{k-1} \phi_j(c) s^j}_{\text{to } \mathrm{D}(c)} + \underbrace{\sum_{j=k+1}^{2k-1} \phi_j(0) s^j}_{\text{to } \mathrm{D}(c^0)}. \tag{16}$$

The term in s^k has been omitted, because it is required only to $\mathrm{D}(c^0)$ yet must be $\mathrm{O}(c)$; in other words, the stage 2(b) transformation that removes the s^k term contributes no $\mathrm{D}(c)$ terms to the normal form, and amounts to simply discarding the s^k term. The stage 2(a) transformation involves only rescaling s by a *constant*.

The only transformation having a non-trivial effect is the tail-removal transformation, which comprises the single iteration

$$T_1\colon\ s = s' + \sum_{i=2}^{k-1} t_i s'^i$$

where the t_i are *constants.* Once the $O(s^{k+2})$ tail has been used to determine T_1, it and all $O(s^k)$ terms produced by applying T_1 to the $D(s^{k-1})$ terms in (16) are simply discarded. The end result of applying T_1 to each s^j is to add a contribution of the form

$$\phi_j(c) \sum_{i=j+1}^{k-1} a_i s^i,$$

where the a_i are constants, which affects all higher powers of s in the unfolding body. Hence T_1 produces the cascade effect:

$$\phi_j(c) \to \phi_j'(c) = \sum_{i=1}^{j-1} d_{ji}\phi_i(c) + \phi_j(c) \qquad \forall 1 \leq j \leq k-1. \tag{17}$$

After including the stage 2(a) rescaling, the dependence of the canonical unfolding variables $\{u_i\}$ on the physical unfolding functions $\{\phi_j\}$ is given by a *lower triangular matrix*:

$$u_i = \sum_{j=1}^{i} m_{ij}\phi_j \tag{18}$$

where $m_{ii} = |\phi_{k+1}(0)|^{-j/(k+1)}$. Such a linear transformation corresponds to a *pure shear* of $\mathbb{R}^{k-1}$. There will be no shearing only in the non-generic case that the $(k-2)$ coefficients $\phi_j(0)$, $k+2 \leq j \leq 2k-1$, all vanish. The only exception is the fold catastrophe, which has $k=2$ and could not manifest a shear anyway since it only has one unfolding variable. The mapping from c_i to $\phi_j(c)$ is responsible for *all* rotation (and relative changes of sense of the axes), and also contributes to the shear.

We conclude this section with a simple example that illustrates shearing 'due to the tail'. Let us analyse the evolute of the plane curve C which is the graph of $y = x^2 + ax^5$. The square of the distance from the point (x, y) on the curve C to a general point (X, Y) is given by

$$\phi(x; X, Y) = (X-x)^2 + (Y - x^2 - ax^5)^2. \tag{19}$$

We will reduce ϕ to linear normal form around the cusp point, which occurs at $x=0$, $X=0$, $Y=\frac{1}{2}$. Hence ϕ is required to $D(x^5)$, and the relevant terms of (19) are simply given by

$$\phi(x{:}\ X, Y) = \phi_0(X, Y) - 2Xx - 2Y'x^2 + x^4 - ax^5, \tag{20}$$

where $Y' \equiv Y - \frac{1}{2}$. The tail-removal transformation is easily found to be $x = x' + \frac{1}{4}ax'^2$, which reduces (20) to the form

$$\phi(x; X, Y) \simeq \phi_0(X, Y) - 2Xx' - (2Y' + aX/2)x'^2 + x'^4.$$

This is exact to linear degree in X and Y', and shows that the cusp is sheared by an amount depending on a, despite the fact that the cusp singularity is 4-determinate; e.g. if $a=4$ the cusp is sheared through 45°.

8. Singularity manifolds to quadratic degree

In the introduction we remarked that our reduction algorithm is not restricted to universal unfoldings, i.e. versal unfoldings of minimal dimension. In this section we consider specifically the case where $\phi(s; c) \equiv \Sigma_{j=0}^{\infty} \phi_j(c)s^j$ is a versal (i.e. stable) but not universal unfolding of an A_k singulaity at $(s; c) = (0; 0) \in \mathbb{R} \times \mathbb{R}^K$, so that $K > k - 1$. Then in fact an A_k singularity occurs at all points of a $(K - k + 1)$-dimensional manifold in $\mathbb{R} \times \mathbb{R}^K$ containing the origin. The projection S_k of this manifold into $\mathbb{R}^K$ is locally smooth; we call it a 'singularity manifold', or more specifically an 'A_k-manifold', e.g. an A_2-surface or an A_3-line in $\mathbb{R}^3$.

The A_k-manifold S_k could be determined by eliminating s from

$$\mathrm{d}^j\phi(s; c)/\mathrm{d}s^j = 0, \qquad 1 \leqslant j \leqslant k,$$

to give $(k-1)$ equations in c, since generally the A_k singularity occurs at a value of s depending on c. However, if ϕ is in normal form (§ 1) then S_k is given directly by

$$u_j(c) = 0, \qquad 1 \leqslant j \leqslant k-1, \tag{21}$$

where $\{u_j(c)\}$ are the $(k-1)$ canonical unfolding functions (cf (2)), because in normal form the main singularity *always* occurs at $s = 0$.

We aim to find a simple prescription for determining the equation of a singularity manifold up to quadratic degree about some point, which we take as the origin. The strategy is first to determine the tangent space of S_k at 0 by considering the reduction of ϕ to linear normal form. This is then used to construct a new coordinate system, if necessary, in which it is clear that most transformations in the reduction of ϕ to quadratic normal form have no affect on the form of S_k up to quadratic degree. For applications of the method described here to the determination of tangent spaces and curvatures see resp. Wright and Dangelmayr (1984) and Dangelmayr and Wright (1984, 1985).

As the diagonal elements of the triangular matrix relating $\{u_i\}$ to $\{\phi_j\}$ in (18) are always non-zero, (21) implies that the equations $\phi_{j,1}(c) = 0$, $1 \leqslant j \leqslant k-1$, determine the tangent space of S_k at 0. The vectors

$$e_j \equiv d\phi_j(0) \in \mathbb{R}^K, \qquad 1 \leqslant j \leqslant k-1,$$

are linearly independent by definition—this is the condition for $\phi(s; c)$ to be a versal unfolding. Therefore, $\{e_j\}_{j=1}^{k-1}$ span the normal space of S_k at 0, and their orthogonal complement in $\mathbb{R}^K$ is its tangent space. To determine curvatures, it is most convenient to have an orthonormal coordinate system related to the tangent space of S_k. From $\{e_j\}_{j=1}^{k-1}$ an orthornormal set $\{\tilde{e}_j\}_{j=1}^{k-1}$ may be constructed by Gram-Schmidt orthonormalisation in the form

$$\tilde{e}_j = \sum_{i=1}^{j} a_{ji} e_i, \qquad 1 \leqslant j \leqslant k-1,$$

where $a_{ii} \neq 0$ and $a_{11} = /|e_1|$ etc. This may then be extended into a complete orthonormal basis $\{\tilde{e}_j\}_{j=1}^{K}$ for $\mathbb{R}^K$. Let $\tilde{c}_j$ be the coordinate along $\tilde{e}_j$, transform ϕ into these new coordinates (if necessary) and drop the tildes, to give

$$\phi_j(c) = \sum_{i=1}^{j} b_{ji} c_i + \mathrm{O}(c^2), \qquad 1 \leqslant j \leqslant k-1, \tag{22}$$

where $b_{ii} \neq 0$.

We define $c_{\mathrm{I}} \equiv (c_1, c_2, \ldots, c_{k-1})$ and $c_{\mathrm{II}} \equiv (c_k, \ldots, c_K)$, which are respectively the coordinates 'normal' and 'tangential' to S_k at 0. Then S_k may be expressed as the graph in $\mathbb{R}^{K-k+1}\times\mathbb{R}^{k-1}$ of

$$c_j=f_j(c_{\mathrm{II}})=\mathrm{O}(c_{\mathrm{II}}^2), \qquad 1\leq j\leq k-1. \tag{23}$$

We seek the quadratic terms of the $f_j(c_{\mathrm{II}})$.

The technique is once again to work backwards, making use of the following crucial observation. Composing (18) with (22) leads to

$$u_i=\sum_{l=1}^{i} M_{il}c_l+\mathrm{O}(c^2), \qquad 1\leq j\leq k-1,$$

where (M_{il}) is once again lower triangular. The last stage of the calculation will be to find the solution of the system (21) in the form (23). Because u_i has the above form, this solution can be performed iteratively, and any $\mathrm{O}(c^2)$ terms involving c_{I} will generate terms $\mathrm{O}(c_{\mathrm{II}}^3)$ in the solution, which are not required. *Hence only terms linear in c_{I} alone and quadratic in c_{II} alone are relevant.*

In the following we shall use this notation:

$$\phi_j \xrightarrow[1]{\text{stage}} \phi_j' \xrightarrow[2]{\text{stage}} \phi_j''.$$

Transformation stage 2(a) only rescales the $u_i(c)$, which has no effect on the solution of (21). Bearing in mind the discussion in § 4, and substituting explicitly for $\alpha(c)$ from (10) into (11), the quadratic terms generated by transformation stage 2(b) are given by

$$\phi_j'(c)\to\phi_j''(c)=\phi_j'(c)-\frac{(j+1)\phi_{j+1,1}'(c)\phi_{k,1}'(c)}{(k+1)\phi_{k+1,0}'} \tag{24a}$$

for $1\leq j\leq k-2$, and

$$\phi_{k-1}'(c)\to\phi_{k-1}''(c)=\phi_{k-1}'(c)-k\phi_{k,1}'^2(c)/2(k+1)\phi_{k+1,0}'. \tag{24b}$$

From (22), $\phi_{j+1,1}'(c)$ in (24*a*) depends *only* on $c_{\mathrm{I}}\equiv\{c_j\}_{j=1}^{k-1}$; hence transformation (24*a*) is irrelevant, because it generates quadratic terms involving c_{I}. The only relevant effect of stage 2 is generated through (24*b*) by those terms in $\phi_k'(c)$ linear in c_{II} alone, because only these generate terms quadratic in c_{II} alone, and is

$$\phi_{k-1}'(c)\to\phi_{k-1}''(c)=\phi_{k-1}'(c)-k\phi_{k,1}'^2(c_{\mathrm{I}}=0,\,c_{\mathrm{II}})/2(k+1)\phi_{k+1,0}'. \tag{25}$$

The effect of transformation stage 1 in general has the form

$$\phi_j(c)\to\phi_j'(c)=\sum_{i=1}^{j-1} d_{ji}(c)\phi_i(c)+\phi_j(c), \qquad j\geq 1 \tag{26}$$

(cf (17) for the linear normal form), which is again a triangular system. As stage 2 has no relevant effect on $\phi_j'(c)$, $1\leq j\leq k-2$, it follows from (26) that

$$u_j(c)=0\Rightarrow\phi_j''(c)\equiv\phi_j'(c)=0\Rightarrow\phi_j(c)=0 \qquad \text{for } 1\leq j\leq k-2,$$

subject to which (26) gives

$$\phi_{k-1}'(c)=\phi_{k-1}(c),$$

$$\phi_{k,1}'(c)=d_{k,k-1}(0)\phi_{k-1,1}(c)+\phi_{k,1}(c),$$

$$\phi_{k+1,0}'=\phi_{k+1,0}.$$

However, only those terms in $\phi'_k(c)$ linear in c_{II} alone are relevant, and from (22) $\phi_{k-1,1}(c)$ has none, so that the transformation of $\phi_k(c)$ is irrelevant. Therefore, stage 1 has no relevant effect whatever, and the tail may be simply discarded.

The prescription. To find S_k up to quadratic degree, which requires only the Taylor coefficients $\{\phi_j(c)\}_{j=1}^{k-1}$ to $\mathrm{D}(c^2)$, $\phi_{k,1}(c)$ and $\phi_{k+1,0}$ of the *original unfolding* ϕ, procede as follows.

(a) Find the tangent space of S_k from $\{\phi_{j,1}(c)\}_{j=1}^{k-1}$, and transform to orthogonal coordinates c_{I} normal to S_k and c_{II} tangential to S_k. This puts $\phi_j(c)$ into the form (cf (22))

$$\phi_j(c) = \sum_{i=1}^{j} b_{ji}c_i + Q_j(c_{\text{II}}) + \mathrm{H}(c_{\text{I}}c_{\text{II}}, c_{\text{I}}^2) + \mathrm{O}(c^3)$$

for $1 \leqslant j \leqslant k-1$, where $Q_j(c_{\text{II}}) = \mathrm{H}(c_{\text{II}}^2)$.

(b) Solve the following triangular system of equations for $c_i(c_{\text{II}}^2)$ iteratively in order of increasing i for $1 \leqslant i \leqslant k-1$:

$$\sum_{i=1}^{j} b_{ji}c_i = -Q_j(c_{\text{II}}) + \delta_{j,k-1} k\phi_{k,1}^2(c_{\text{I}}=0, c_{\text{II}})/2(k+1)\phi_{k+1,0}, \tag{27}$$

where $b_{ii} \neq 0$ and $\delta_{j,k-1}$ is the Kronecker symbol.

We conclude with two examples, intended to illustrate respectively the change of coordinates and a physical application. First suppose

$$\phi_1(c) = 2^{-1/2}(c_1 - c_2) + \gamma_1 c_3^2 + \ldots, \qquad \phi_2(c) = 2^{1/2}(c_1 + 2c_2) + \gamma_2 c_3^2 + \ldots,$$

$$\phi_3(c) = \alpha c_1 + \beta c_2 + \gamma_3 c_3 + \mathrm{O}(c^2), \qquad \phi_4(c) = \tfrac{1}{4} + \mathrm{O}(c),$$

where ... means terms $\mathrm{O}(c^2)$ other than c_3^2. Then $\phi(s; c)$ has an A_3 singularity at the origin, i.e. a rib passes smoothly through $0 \in \mathbb{R}^3$. The normal coordinates are $c_{\text{I}} \equiv (\tilde{c}_1, \tilde{c}_2)$ where $\tilde{c}_1 = 2^{-1/2}(c_1 - c_2)$ and $\tilde{c}_2 = 2^{-1/2}(c_1 + c_2)$. The tangent coordinate is $c_{\text{II}} \equiv \tilde{c}_3 = c_3$. Then

$$\tilde{\phi}_1(\tilde{c}) = \tilde{c}_1 + \gamma_1 \tilde{c}_3^2 + \ldots, \qquad \tilde{\phi}_2(\tilde{c}) = -\tilde{c}_1 + 3\tilde{c}_2 + \gamma_2 \tilde{c}_3^2 + \ldots,$$

$$\tilde{\phi}_3(\tilde{c}) = \tilde{\alpha}\tilde{c}_1 + \tilde{\beta}\tilde{c}_2 + \gamma_3\tilde{c}_3 + \mathrm{O}(\tilde{c}^2)$$

and equations (27) become

$$\tilde{c}_1 = -\gamma_1 \tilde{c}_3^2, \qquad -\tilde{c}_1 + 3\tilde{c}_2 = -\gamma_2 \tilde{c}_3^2 - \tfrac{3}{2}\gamma_3^2 \tilde{c}_3^2.$$

Therefore, the equation of the rib in the original coordinates is

$$c_1 \equiv 2^{-1/2}(\tilde{c}_2 + \tilde{c}_1) = 2^{-1/2}[\tfrac{1}{2}\gamma_3^2 - \tfrac{1}{3}(4\gamma_1 + \gamma_2)]c_3^2 + \mathrm{O}(c_3^3),$$

$$c_2 \equiv 2^{-1/2}(\tilde{c}_2 - \tilde{c}_1) = 2^{-1/2}[\tfrac{1}{2}\gamma_3^2 + \tfrac{1}{3}(2\gamma_1 - \gamma_2)]c_3^2 + \mathrm{O}(c_3^3).$$

Our second example is related to that used in § 7, and involves a 'bivariate catastrophe'. Consider the evolute of the surface whose height in cartesian coordinates is

$$h(x, y) = x^2 + ay^2, \qquad a \neq 1.$$

It has ribs cutting the z axis at $z = \frac{1}{2}$ and $z = 1/2a$. We will find the curvature at the z axis of the rib that cuts it at $z = \frac{1}{2}$. The squared distance $\phi(x, y; X, Y, Z)$ from the point $(x, y, h(x, y))$ on the surface to (X, Y, Z) is given by

$$\begin{aligned}\phi &= (X-x)^2+(Y-y)^2+(Z-x^2-ay^2)^2 \\ &= \phi_0 - 2(Xx+Yy+Z'x^2)+\eta y^2+x^4+2ax^2y^2+a^2y^4\end{aligned}$$

where $Z' \equiv Z - \frac{1}{2}$ and $\eta \equiv 1-2aZ = 1-a-2aZ'$. The rib is obviously tangent to the y axis, so $c_{\mathrm{I}} \equiv (X, Z)$ and $c_{\mathrm{II}} \equiv Y$. Therefore we need ϕ_1 and ϕ_2 only to linear degree in X and Z and quadratic degree in Y, and ϕ_3 only to linear degree in Y.

As there are two state variables, we must first apply the splitting lemma, which amounts to reducing the y-dependence of ϕ to normal form up to quadratic degree in its control variable Y. The y singularity is Morse or A_1, so ϕ is required only up to $\mathrm{D}(y^2)$ (because $k=1$, $m=2$ gives $(m+1)k-1=2$, see § 5). Therefore, the relevant y-dependence of ϕ is $-2Yy+Ay^2$, where $A \equiv A(x) \equiv \eta + 2ax^2$. In normal form this is simply $y'^2 - Y^2/A$, where $y' \equiv A^{1/2}(y - Y/A)$, which is locally well defined because $A(0) \neq 0$. Then using the expansion $1/A(x) = 1/\eta - 2ax^2/\eta^2 + \mathrm{O}(x^4)$, the splitting lemma leads to

$$\phi = (\phi_0 - Y^2/\eta) - 2Xx + 2(aY^2/\eta^2 - Z')x^2 + \mathrm{O}(x^4) + y'^2.$$

Note that because there is no relevant ϕ_3 term, $\phi_{4,0}$ is not required either. Recalling that $\eta \equiv 1-a-2aZ'$, the rib is given to second degree by

$$X = 0, \qquad Z' = aY^2/(1-a)^2.$$

This is clearly correct for $a=0$, and gives infinite curvature in the limit $a \to 1$. This is consistent with the fact that in this limit $h(x, y)$ is rotationally symmetric, and therefore so must its evolute be. In fact, this situation gives a highly unstable infinite codimension umbilic singularity (see figure A2.3 of Berry and Upstill (1980), figure 41 of Berry (1981), figure 21 of Bruce *et al* (1984)).

Acknowledgments

We thank Karl Millington for his indirect contribution to this work via collaboration with FJW and for suggesting improvements to the manuscript. The work was supported by the Stiftung Volkswagenwerk, FRG, which also supported in part KM's visits to the Institut. We thank Professor J F Nye for a preprint of the paper by Nye and Hannay (1984), which has been a catalyst for our own work.

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